Modeling a Linear Relationship

Lecture 45

Sections 13.1 - 13.3.1

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Outline

1. Introduction
2. Scatterplots
3. Describing Relationships
4. Linear Regression
5. Linear Regression
   - Which Line is Better?
   - Measuring the Goodness of Fit
6. Summary

Introduction

- In Chapter 14, we investigated the relationship between two or more qualitative variables.
- The basic question was, are the variables independent?
- Now in Chapter 13, we will investigate the relationship between two quantitative variables.
- Because the variables are quantitative rather than qualitative, the basic problem will be to give a quantitative description of their relationship.
- Through this description, we hope to be able to predict the value of one variable when we know the value of the other.

Bivariate Data

Definition (Bivariate)

Data are called bivariate if each two observations, which we will call x and y, are made for each member of the sample.

- x is the explanatory variable.
- y is the response variable.
- x is also called the independent variable.
- y is also called the dependent variable.

Scatterplots

Definition (Scatterplot)

A display in which each observation (x, y) is plotted as a point in the xy-plane.

Free Lunches vs. Graduation Rates

District	Free Lunch	Grad. Rate
Amelia	41.2	68.9
Caroline	40.2	62.9
Charles City	45.8	67.7
Chesterfield	22.5	80.5
Colonial Hgts	25.7	73.0
Cumberland	55.3	63.9
Dinwiddie	45.2	71.4
Goochland	23.3	76.3
Hanover	13.7	90.1
Henrico	30.2	81.1
Hopewell	63.1	63.4

Free Lunches vs. Graduation Rates

District	Free Lunch	Grad. Rate
King and Queen	59.9	64.1
King William	27.9	67.0
Louisa	44.9	80.1
New Kent	13.9	77.0
Petersburg	61.6	54.6
Powhatan	12.2	89.3
Prince George	30.9	85.0
Richmond	74.0	46.9
Sussex	74.8	59.0
West Point	19.1	82.0

Scatter Plot

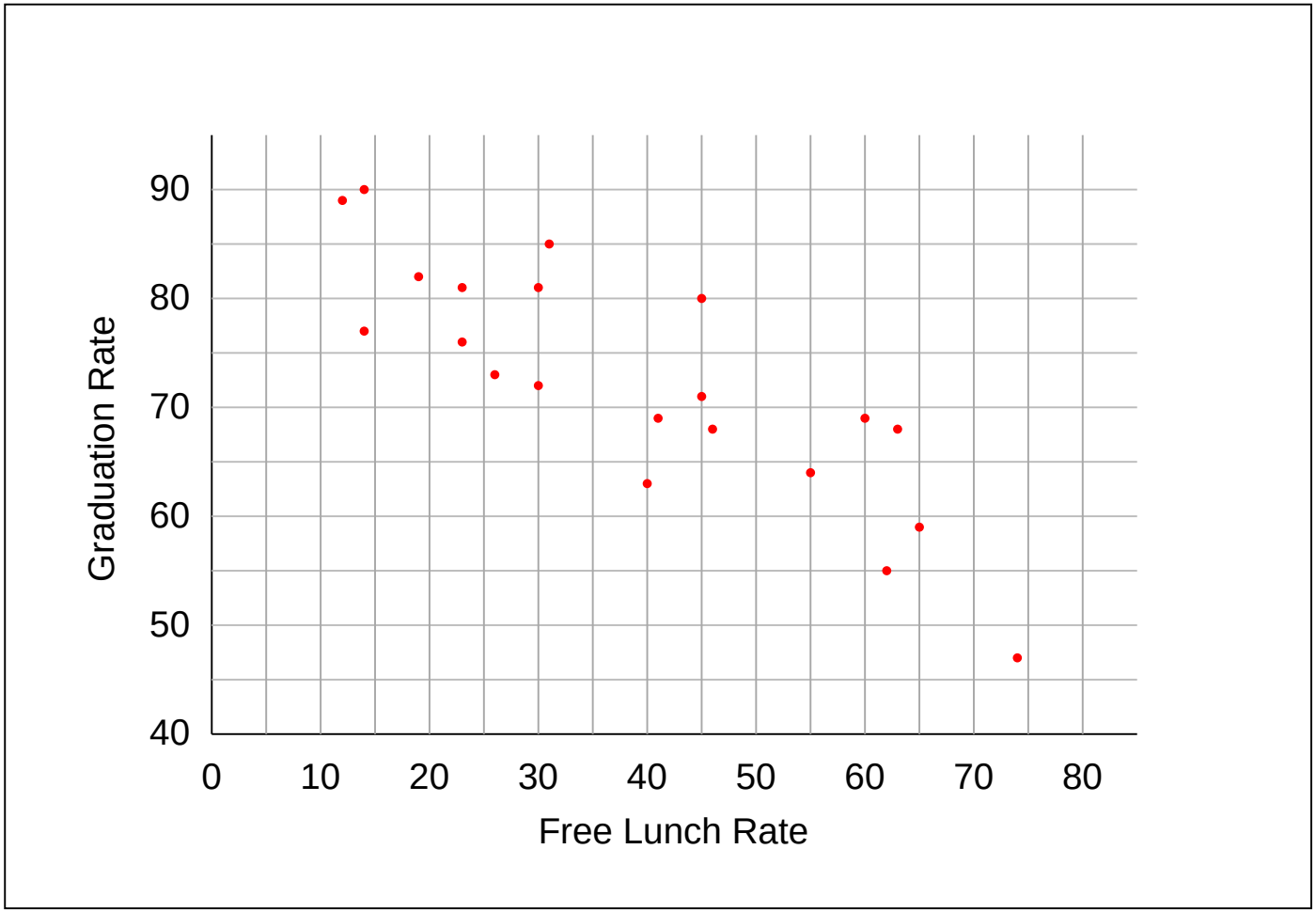
Describing a Relationship

- Does there appear to be a relationship?
- How can we tell?
- How would we describe the relationship?

Linear Association

- Draw (or imagine) an oval around the data set.
- If the oval is tilted, then there is some linear association.
- If the oval is tilted upwards from left to right, then there is positive association.
- If the oval is tilted downwards from left to right, then there is negative association.
- If the oval is not tilted at all, then there is no association.

Free-Lunch Participation vs. Graduation Rate



Free-Lunch Participation vs. Graduation Rate

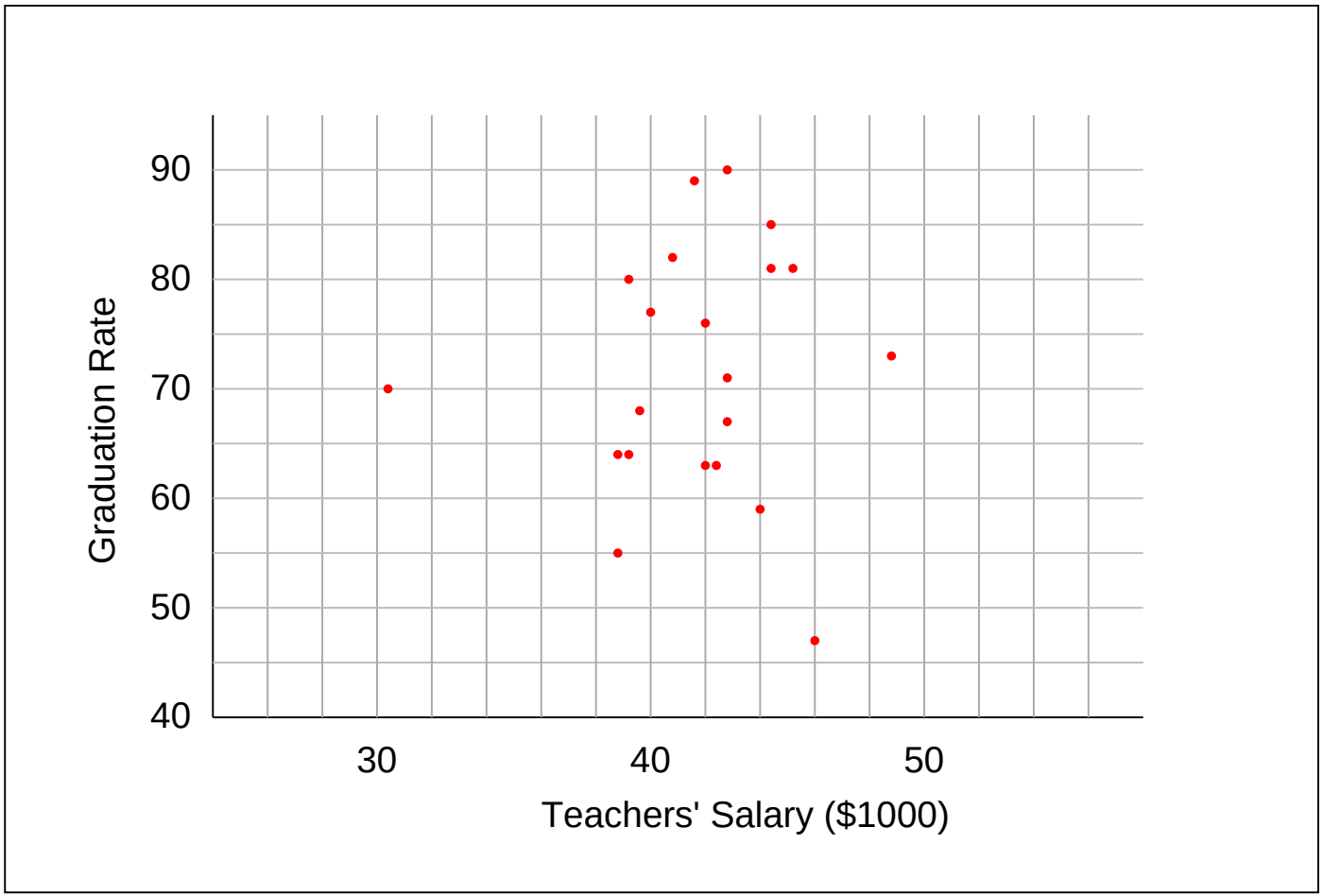
Teachers' Salary vs. Graduation Rate

District	Avg. Salary	Grad. Rate
Amelia	30446	68.9
Caroline	41935	62.9
Charles City	39530	67.7
Chesterfield	44417	80.5
Colonial Hgts	48999	73.0
Cumberland	39380	63.9
Dinwiddie	42866	71.4
Goochland	41893	76.3
Hanover	42715	90.1
Henrico	45021	81.1
Hopewell	42351	63.4

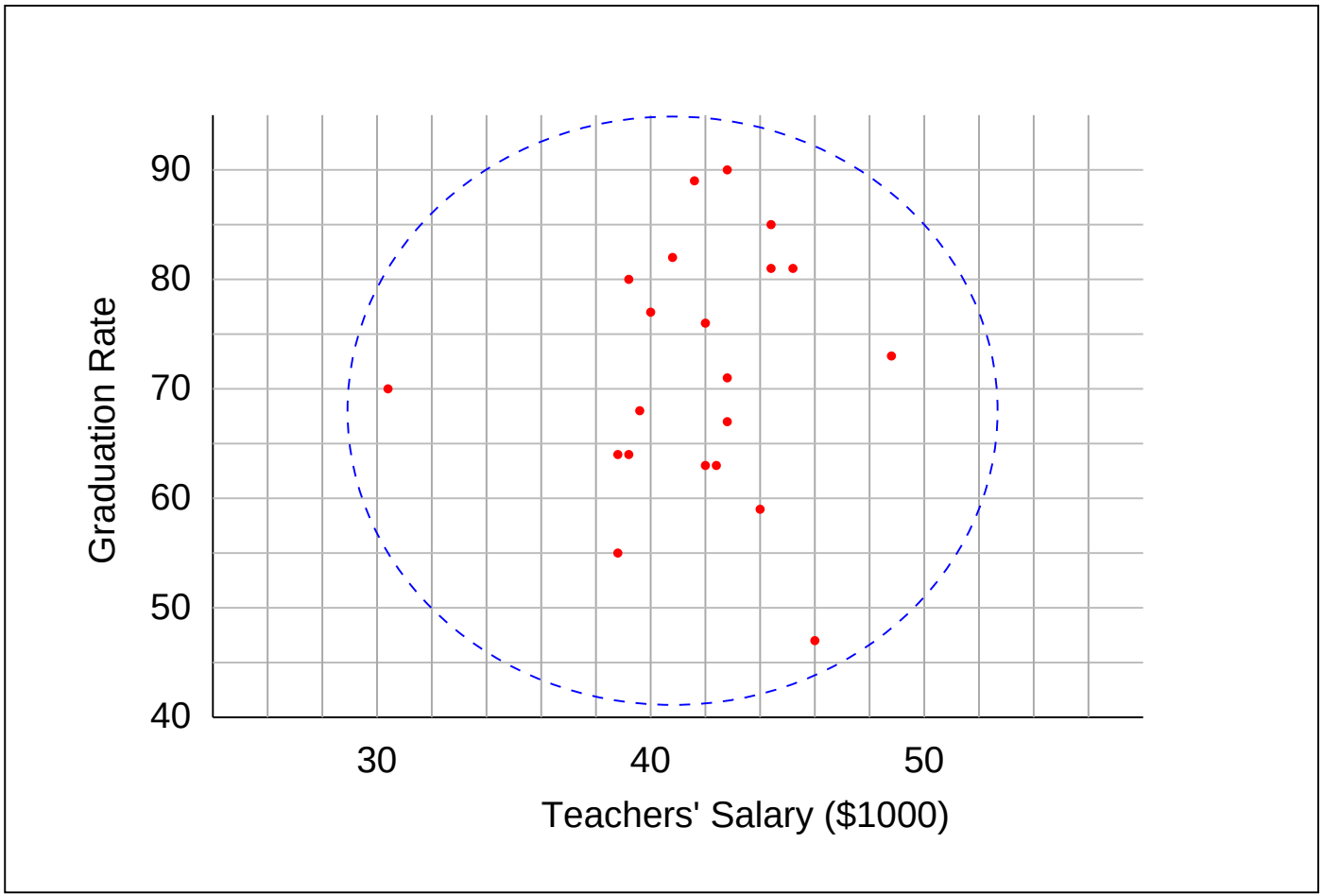
Teachers' Salary vs. Graduation Rate

District	Avg. Salary	Grad. Rate
King and Queen	38803	64.1
King William	42750	67.0
Louisa	39010	80.1
New Kent	39891	77.0
Petersburg	38252	54.6
Powhatan	41523	89.3
Prince George	44529	85.0
Richmond	45875	46.9
Sussex	44142	59.0
West Point	40797	82.0

Teachers' Salary vs. Graduation Rate



Teachers' Salary vs. Graduation Rate



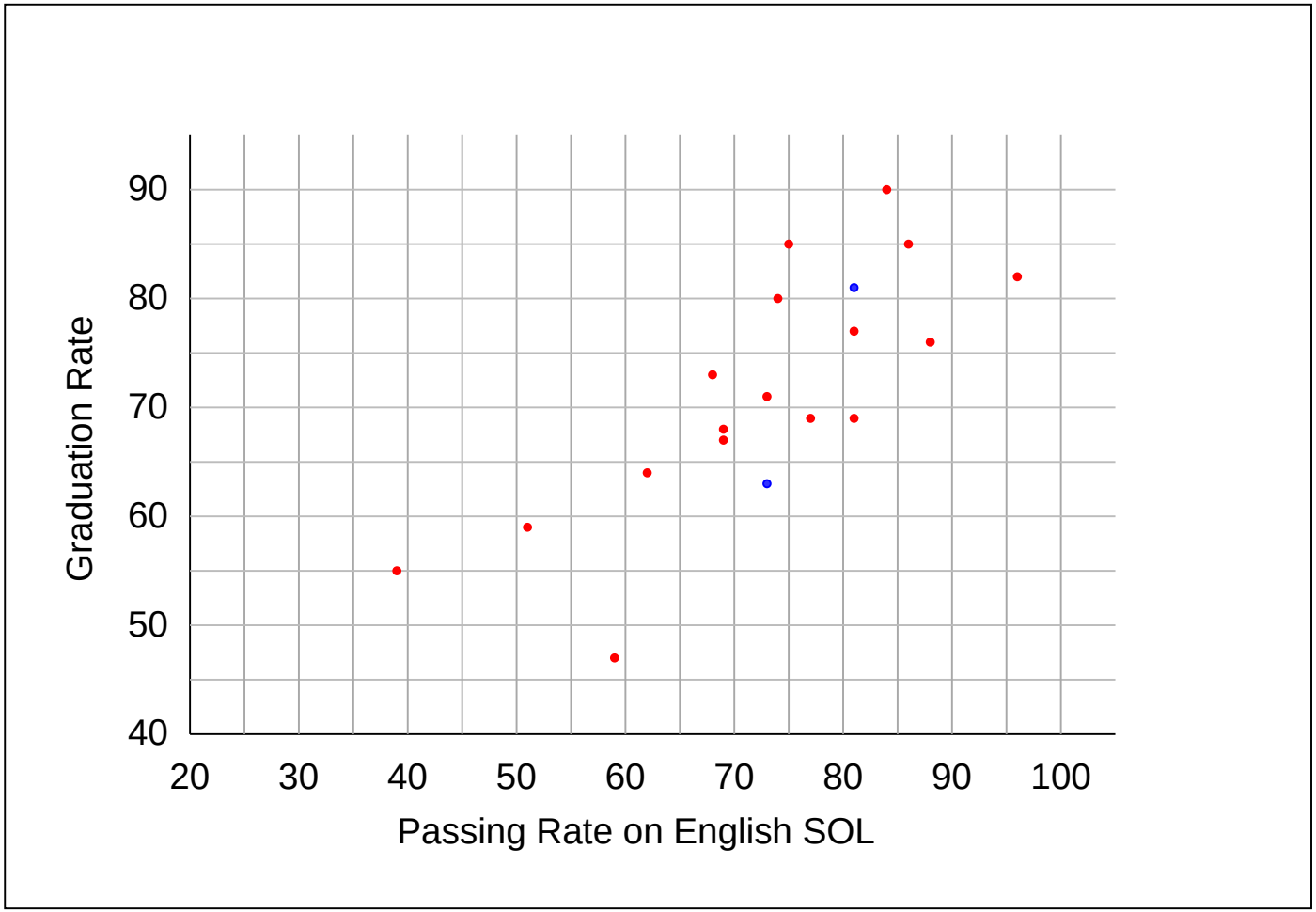
Passing Rate on English SOL vs. Graduation Rate

District	SOL Rate	Grad. Rate
Amelia	77	68.9
Caroline	73	62.9
Charles City	69	67.7
Chesterfield	81	80.5
Colonial Hgts	68	73.0
Cumberland	81	63.9
Dinwiddie	73	71.4
Goochland	88	76.3
Hanover	84	90.1
Henrico	81	81.1
Hopewell	73	63.4

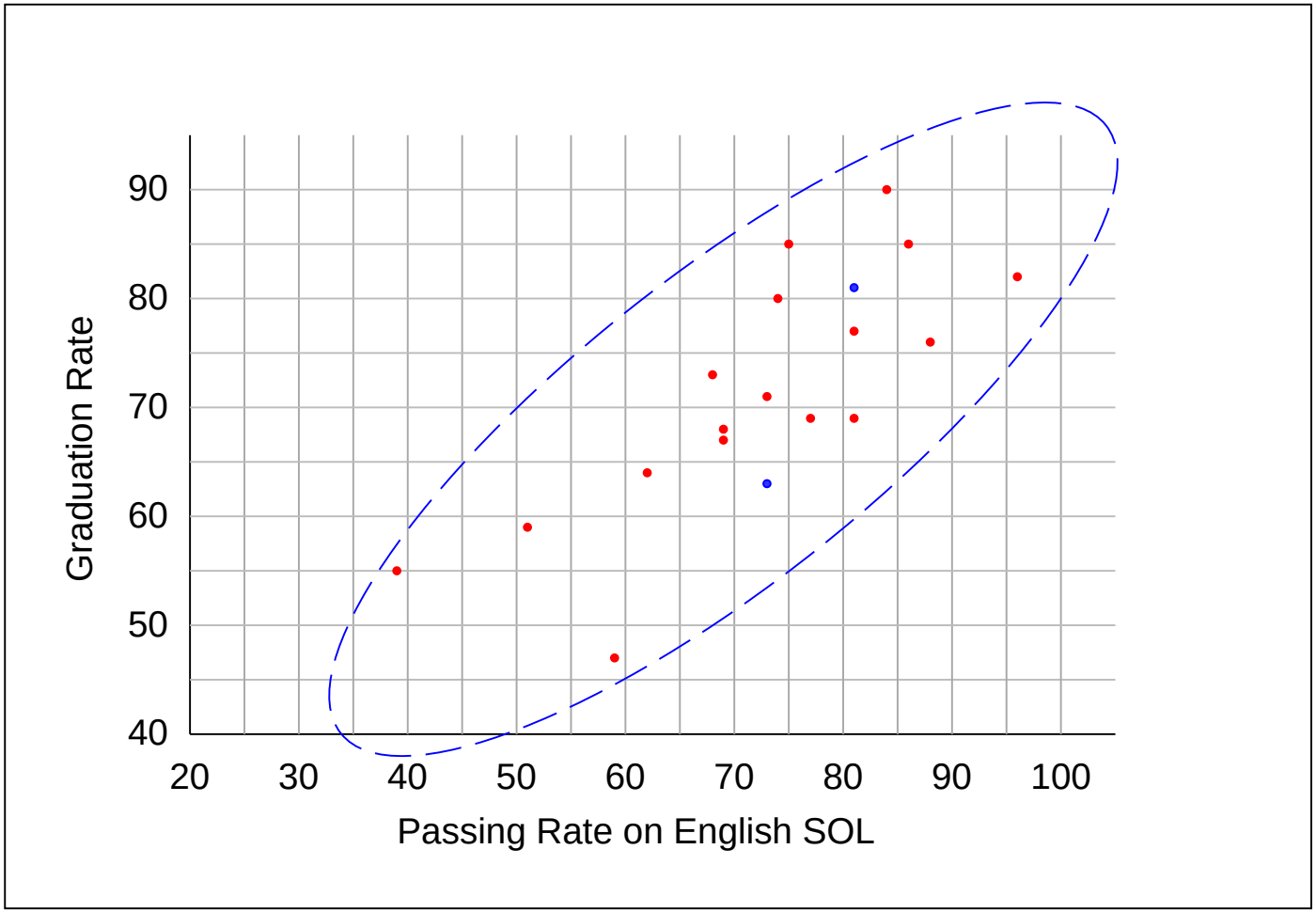
Passing Rate on English SOL vs. Graduation Rate

District	SOL Rate	Grad. Rate
King and Queen	62	64.1
King William	69	67.0
Louisa	74	80.1
New Kent	81	77.0
Petersburg	39	54.6
Powhatan	86	89.3
Prince George	75	85.0
Richmond	59	46.9
Sussex	51	59.0
West Point	96	82.0

Passing Rate on English SOL vs. Graduation Rate



Passing Rate on English SOL vs. Graduation Rate



Strong vs. Weak Association

- The association is strong if the oval is narrow.
- The association is weak if the oval is wide.

Example

- Draw a scatterplot of the following data.

x	y
1	8
3	12
4	9
5	14
8	16
9	20
11	17
15	24

Scatterplots on the TI-83

- To set up a scatterplot,
  - Enter the x values in $\texttt{L}_1$.
  - Enter the y values in $\texttt{L}_2$.
  - Press `2nd STAT PLOT`.
  - Select `Plot1` and press `ENTER`. The `Stat Plot` display appears.

TI-83 - Scatterplots

- In the Stat Plot display,
  - Select `On` and press `ENTER`.
  - Under `Type`, select the first icon (a small image of a scatterplot) and press `ENTER`.
  - For `XList`, enter L_1.
  - For `YList`, enter L_2.
  - For `Mark`, select the one you want and press `ENTER`.

TI-83 - Scatterplots

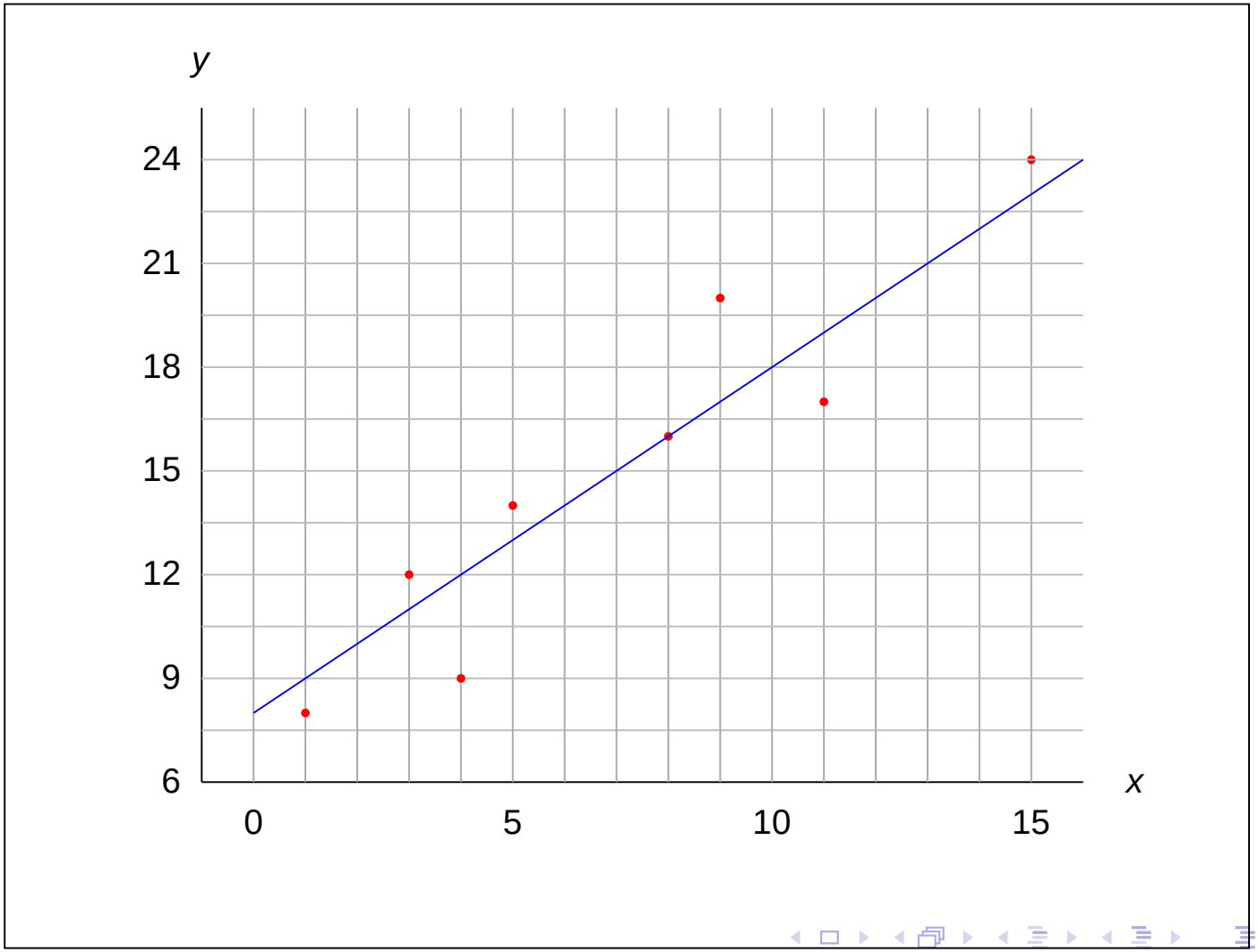
- To draw the scatterplot,
  - Press `ZOOM`. The `Zoom` menu appears.
  - Select `ZoomStat` (#9) and press `ENTER`. The scatterplot appears.
  - Press `TRACE` and use the arrow keys to inspect the individual points.

Simple Linear Regression

- To quantify the linear relationship between x and y, we wish to find the equation of the line that "best" fits the data.
- Typically, there will be many lines that all look pretty good.
- How do we measure how well a line fits the data?

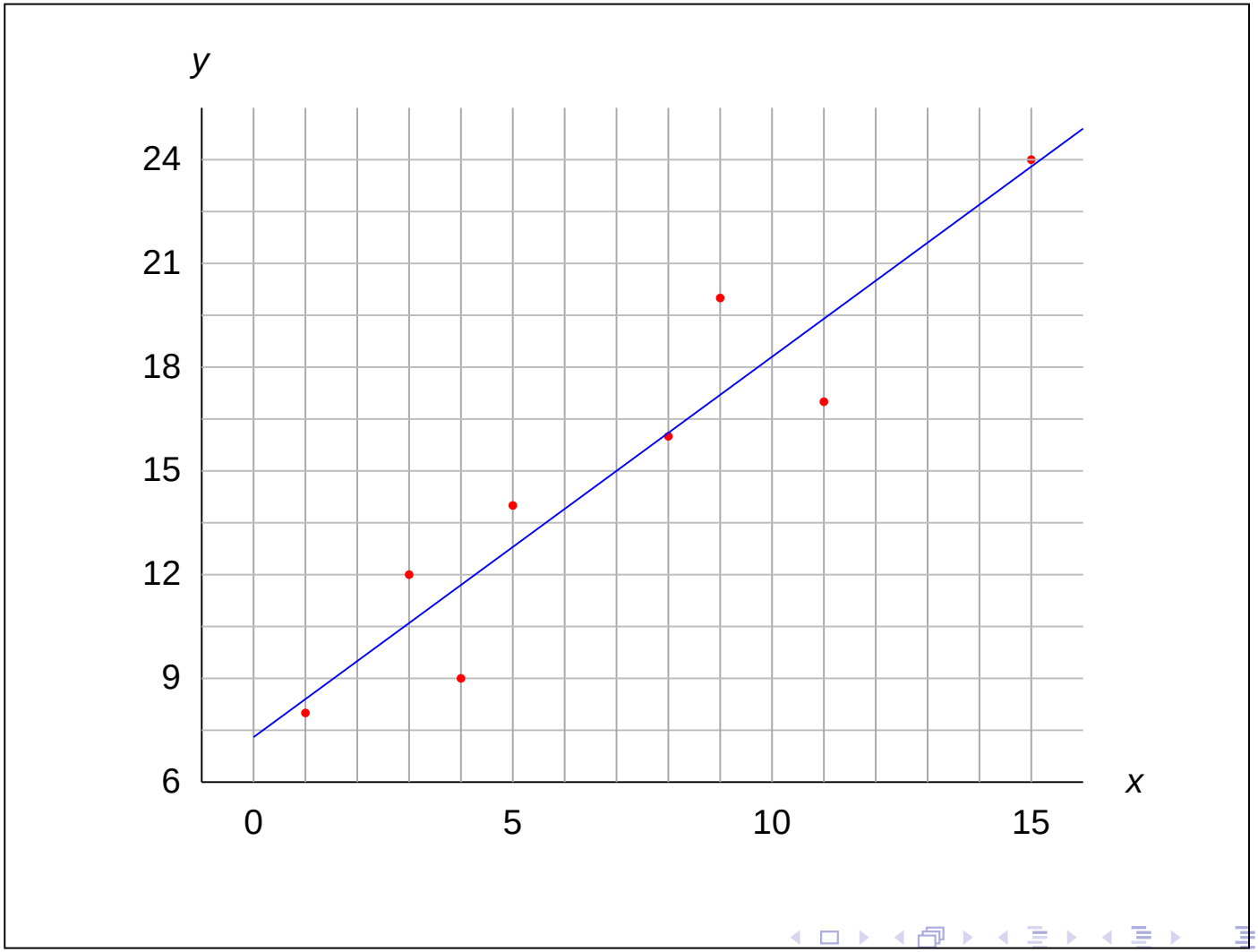
Simple Linear Regression

- Which line better fits the data?



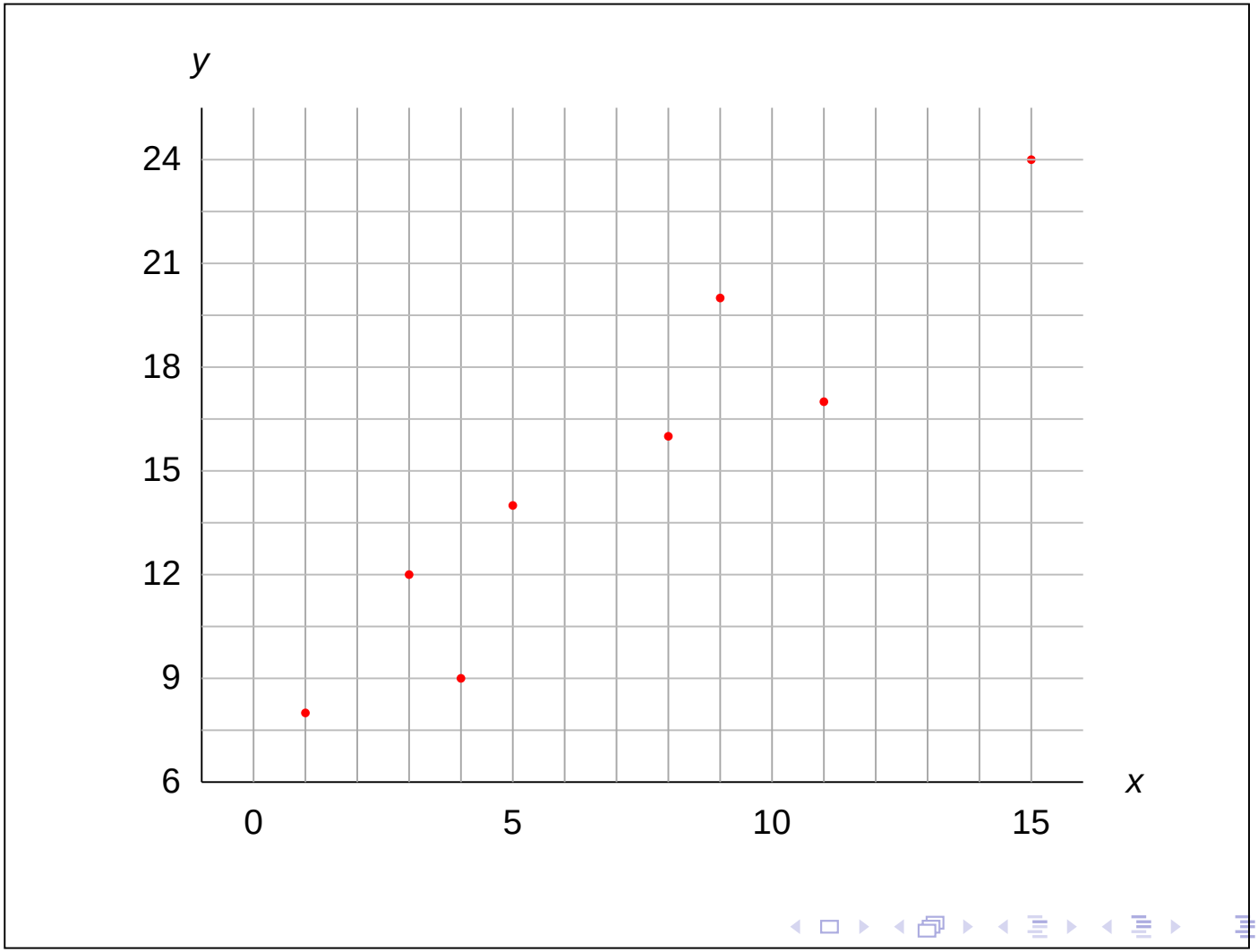
Simple Linear Regression

- Which line better fits the data?



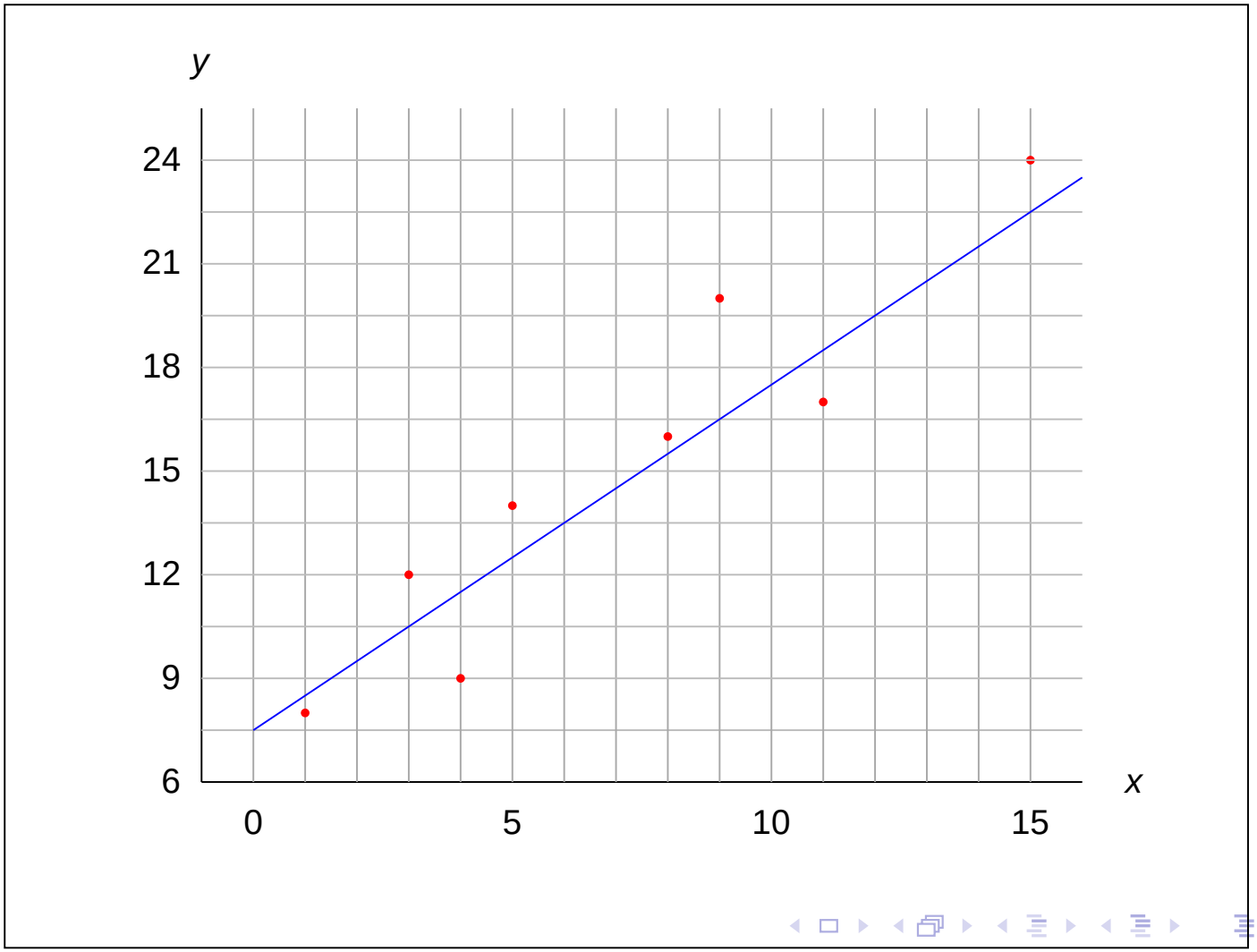
Measuring the Goodness of Fit

- Start with the scatterplot.



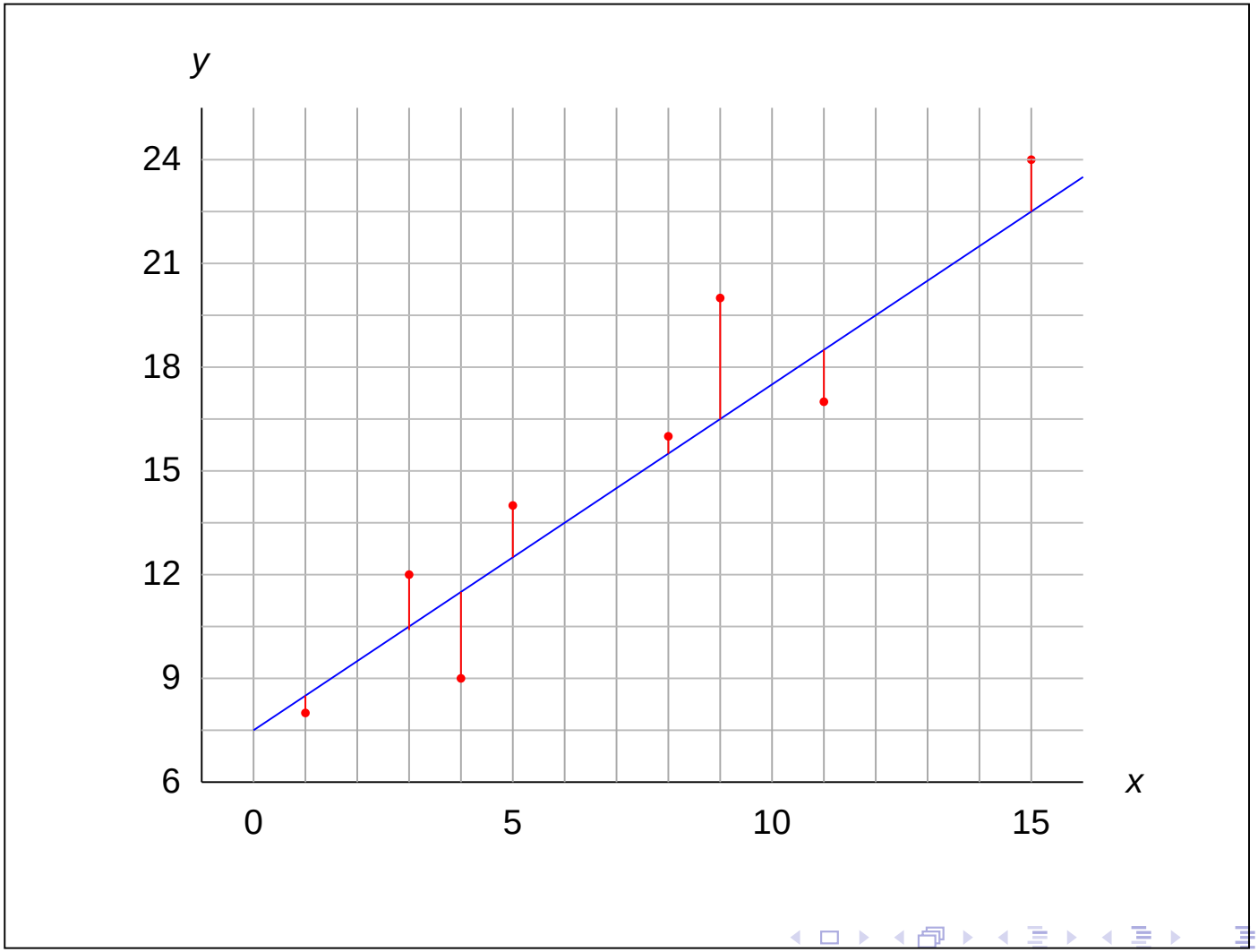
Measuring the Goodness of Fit

- Draw any line through the scatterplot.



Measuring the Goodness of Fit

- Measure the vertical distances to the line.



Residuals

- The vertical distances are called the residuals.

Definition (Residual)

The i^{th} residual is the difference between y_i and $\hat{y}_i$.

- The formula for the i^{th} residual is

$$e_i = y_i - \hat{y}_i.$$

where y_i is the observed value and $\hat{y}_i$ is the value predicted by the model.

Measuring the Goodness of Fit

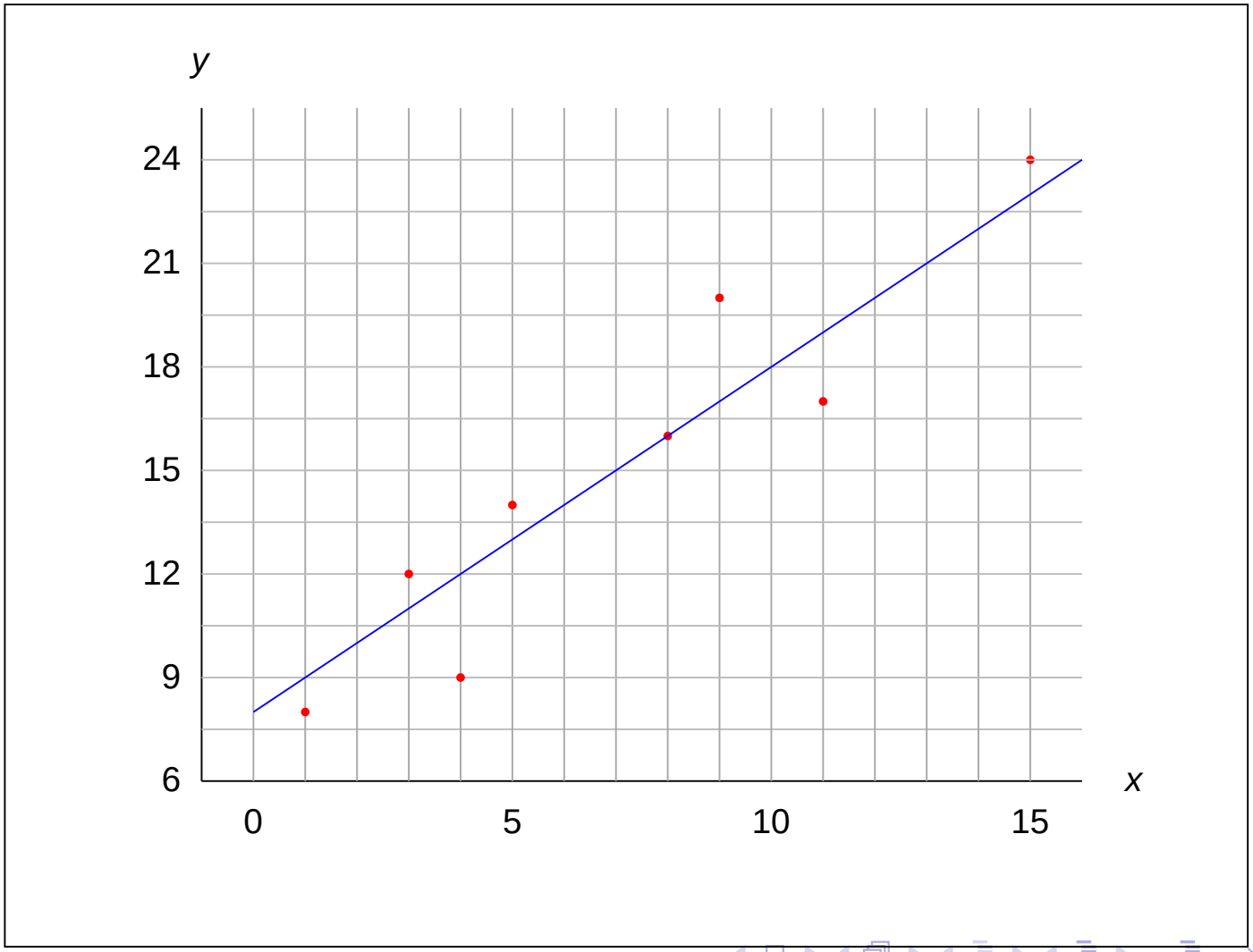
Definition (Line of Best Fit)

The line of best fit is the line with the smallest possible sum of squared residuals.

- The line of best fit is also called the least squares line and the regression line.

Least Squares Line

- Let's see how good the fit is for the line $\hat{y} = 8 + x$.



Example

- Start with the data points

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8			
3	12			
4	9			
5	14			
8	16			
9	20			
11	17			
15	24			

Example

- Compute the predicted y, using $\hat{y} = 8 + x$.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	9		
3	12	11		
4	9	12		
5	14	13		
8	16	16		
9	20	17		
11	17	19		
15	24	23		

Example

- Find the residues.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	9	-1	
3	12	11	1	
4	9	12	-3	
5	14	13	1	
8	16	16	0	
9	20	17	3	
11	17	19	-2	
15	24	23	1	

Example

- Square the residues.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	9	-1	1
3	12	11	1	1
4	9	12	-3	9
5	14	13	1	1
8	16	16	0	0
9	20	17	3	9
11	17	19	-2	4
15	24	23	1	1

Example

- Add up the residues.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	9	-1	1
3	12	11	1	1
4	9	12	-3	9
5	14	13	1	1
8	16	16	0	0
9	20	17	3	9
11	17	19	-2	4
15	24	23	1	1
				26

Example

- The sum of the squared residues is called the sum of squared errors (SSE).

$$\text{SSE} = \sum (y - \hat{y})^2 = 1 + 1 + 9 + 1 + 0 + 9 + 4 + 1 = 26.$$

Computing Residuals on the TI-83

- We can compute the residuals and SSE on the TI-83.
  - Enter the x-values in list $\texttt{L}_1$ and the y-values in list $\texttt{L}_2$.
  - Compute $a + b*\texttt{L}_1$ and store in list $\texttt{L}_3$ ($\hat{y}$ values).
  - Compute $(\texttt{L}_2-\texttt{L}_3)^2$. This is a list of the squared residuals.
  - Compute $\texttt{sum(Ans)}$. This is the sum of the squared residuals.

Sum of Squared Residuals

- Let's see how good the fit is for the line

$$\hat{y} = 7.3 + 1.1x$$

.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8			
3	12			
4	9			
5	14			
8	16			
9	20			
11	17			
15	24			

Sum of Squared Residuals

- Let's see how good the fit is for the line

$$\hat{y} = 7.3 + 1.1x$$

.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	8.4		
3	12	10.6		
4	9	11.7		
5	14	12.8		
8	16	16.1		
9	20	17.2		
11	17	19.4		
15	24	23.8		

Sum of Squared Residuals

- Let's see how good the fit is for the line

$$\hat{y} = 7.3 + 1.1x$$

.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	8.4	-0.4	
3	12	10.6	1.4	
4	9	11.7	-2.7	
5	14	12.8	1.2	
8	16	16.1	-0.1	
9	20	17.2	2.8	
11	17	19.4	-2.4	
15	24	23.8	0.2	

Sum of Squared Residuals

- Let's see how good the fit is for the line

$$\hat{y} = 7.3 + 1.1x$$

.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	8.4	-0.4	0.16
3	12	10.6	1.4	1.96
4	9	11.7	-2.7	7.29
5	14	12.8	1.2	1.44
8	16	16.1	-0.1	0.01
9	20	17.2	2.8	7.84
11	17	19.4	-2.4	5.76
15	24	23.8	0.2	0.04

Sum of Squared Residuals

- Let's see how good the fit is for the line

$$\hat{y} = 7.3 + 1.1x$$

.

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})^2$
1	8	8.4	-0.4	0.16
3	12	10.6	1.4	1.96
4	9	11.7	-2.7	7.29
5	14	12.8	1.2	1.44
8	16	16.1	-0.1	0.01
9	20	17.2	2.8	7.84
11	17	19.4	-2.4	5.76
15	24	23.8	0.2	0.04
				24.50

Sum of Squared Residuals

- We conclude that $\hat{y} = 7.3 + 1.1x$ is a better fit than $\hat{y} = 8 + x$.
- Is it the best fit?
- For all the lines that one could draw through this data set, it turns out that 24.50 is the smallest possible value for the sum of the squares of the residuals.
- Therefore,

$$\hat{y} = 7.3 + 1.1x$$

  is the regression line for this data set.

Prediction

Definition (Interpolation)

Using an x value within the observed extremes of x values to predict y.

Definition (Extrapolation)

Using an x value beyond the observed extremes of x values to predict y.

- Interpolated values are more reliable then extrapolated values.
- The farther out the values are extrapolated, the less reliable they are.

Interpolation vs. Extrapolation

- Use the regression line to predict y when
- $x = 6$
- $x = 12$
- $x = 30$

Interpolation vs. Extrapolation

- Use the regression line to predict y when
- $x = 6$: $\hat{y}(6) = 7.3 + 1.1(6) = 13.9$.
- $x = 12$: $\hat{y}(12) = 7.3 + 1.1(12) = 20.5$.
- $x = 30$: $\hat{y}(30) = 7.3 + 1.1(30) = 40.3$.

Summary

Summary

- We can gain a lot of insight into the relationship between two variables by viewing a scatterplot.
- However, the scatterplot alone will not allow us to use one variable to predict the value of the other.
- Of all possible lines, the regression line is the line that best fits the data.
- By using the regression equation, we can predict y when we know x.